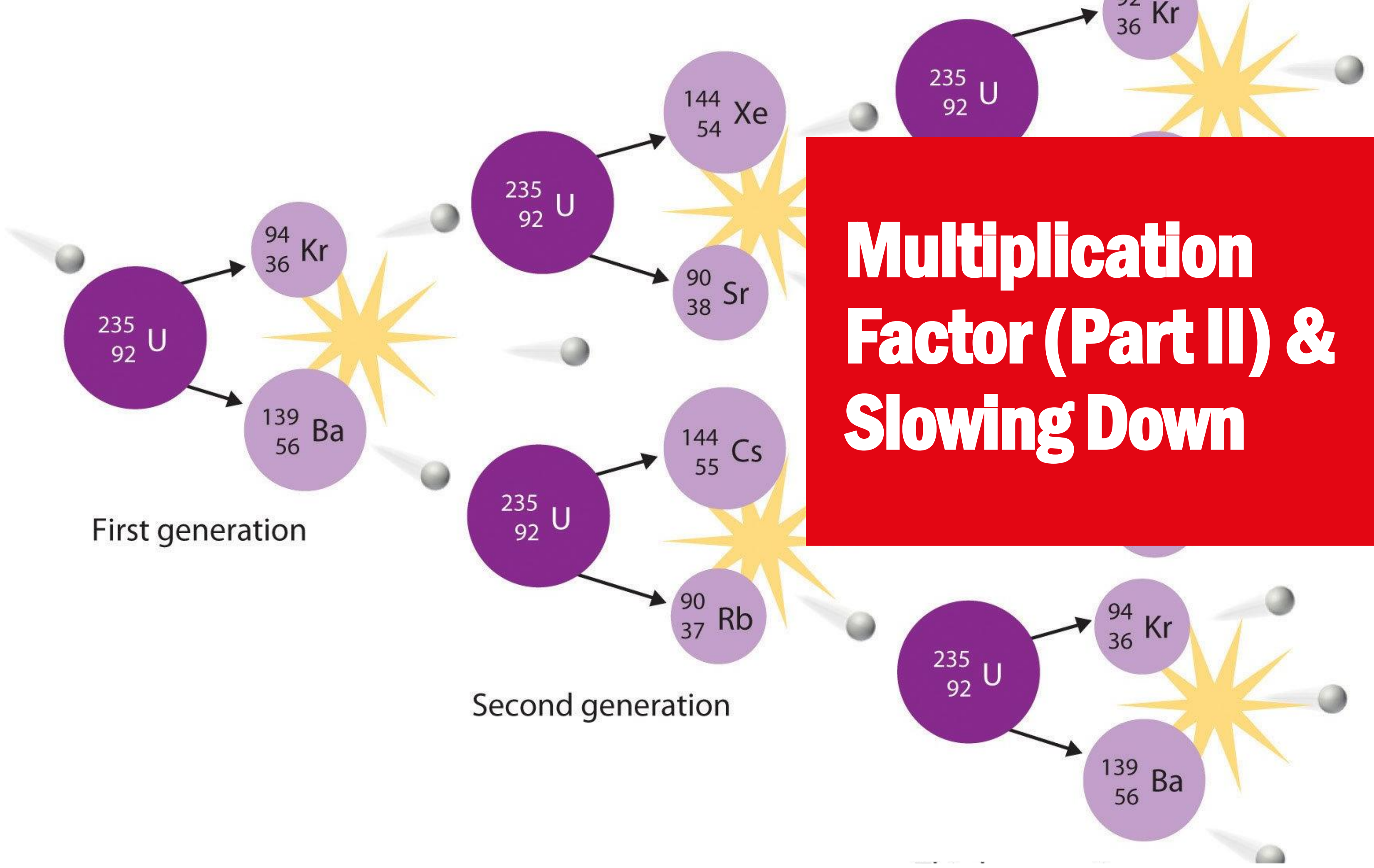
Multiplication Factor (Part II) & Slowing Down



Reminder on Multiplication Factor

- For a reactor with neutron production rate P, absorption rate (A) and leakages (L), we can define the multiplication factor as:

$$\mathbf{k = Multiplication\ Factor} = \frac{\text{The number of fissions in generation i}}{\text{The number of fissions in previous generation (i − 1)}} = \frac{P}{A + L}$$

- If one fission leads to more than one fission the fission rate increases in time exponentially
  - Such a reactor is called **supercritical and k >1**
- If one fission leads to less than one fission the fission rate decreases in time and eventually the chain reaction stops
  - Such a reactor is called **subcritical and k < 1**
- If one fission leads on average to exactly one other fission the chain reaction is self-sustained.
  - Such reactor is **called critical and k =1**

Critical Reactors – One-group Reactor Equation

- For a critical infinite system:

$$k_\infty = \frac{P}{A} = \frac{\bar{\nu}\Sigma_f}{\Sigma_a} = \frac{\bar{\nu}\Sigma_f}{\Sigma_{a,fuel} + \Sigma_{a,nonFuel}} = \frac{\bar{\nu}\Sigma_f}{\Sigma_{a,fuel}} \cdot \frac{\Sigma_{a,fuel}}{\Sigma_{a,fuel} + \Sigma_{a,nonFuel}} = \eta \cdot f = 1$$

where **f is the fuel utilization factor**, i.e. how many neutrons are absorbed in the fuel per total absorption in the system.

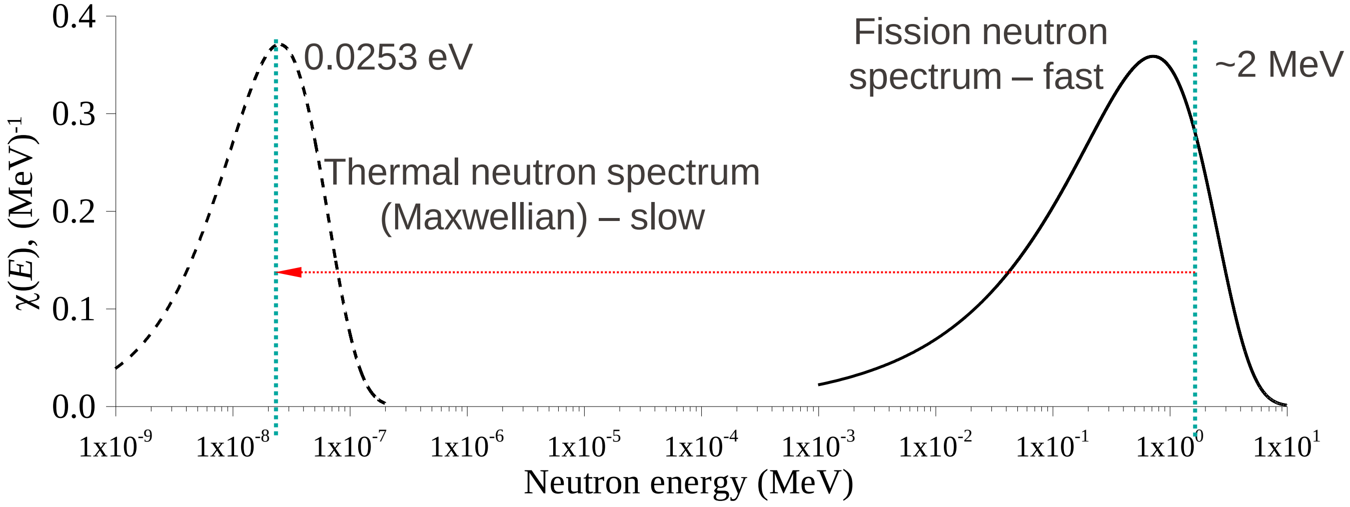
- In finite systems some neutrons escape, so $k_\infty = 1$ is not enough. One talks about k-effective

$$k_{eff} = \frac{P}{A + L} = \frac{P}{A}\frac{A}{A + L} = k_\infty \cdot P_{NL} = 1$$

where P_{NL} **is the non-leakage probability**, i.e. the probability that one neutron remains in the system without leaking and is finally absorbed in the system (parasitic or fissile).

Fission spectrum ⟶ Thermal spectrum

- For a thermal reactor, neutrons are born fast and are absorbed in thermal range.
- "Slowing down factor" of almost 10^8! (From approx. 2 MeV to 0.0253 eV)
- Moderators needed (light nuclei: H2O, graphite,…)



Infinite Thermal Reactor – 4 factor formula

For a thermal reactor one can express the multiplication factor in a convenient form:

$$k_\infty = \frac{P}{A} = \frac{P}{P_{th}} \times \frac{P_{th}}{A_{th}^{fuel}} \times \frac{A_{th}^{fuel}}{A_{th}} \times \frac{A_{th}}{A} = \varepsilon \times \eta_{th} \times f \times p$$

$$\varepsilon = \frac{P}{P_{th}} = \frac{P_{th} + P_{fast}}{P_{th}} = 1 + \frac{P_{fast}}{P_{th}}$$

→ **Fast fission factor:** bonus from fast fissions in non-fissile material. Typical value ~ 1.04

$$\eta_{th} = \frac{P_{th}}{A_{th}^{fuel}} < \nu$$

→ **Eta or (thermal) fission factor**: # of n emitted per thermal neutron absorbed in fuel. Typical value ~ 2.02

$$f = \frac{A_{th}^{fuel}}{A_{th}} = \frac{A_{th}^{fuel}}{A_{th}^{fuel} + A_{th}^{nonFuel}}$$

→ **(Thermal) Fuel utilization factor:** fraction of thermal neutrons asorbed *in fuel.* Typical value ~ .87

$$p = \frac{A_{th}}{A} = \frac{A - A_{reson}}{A} = 1 - \frac{A_{reson}}{A}$$

→ **Resonance escape probability** = fraction of n absorbed i*n thermal region.* Typical value ~ 0.8

4 factor formula – Thermal Multiplication Factor

$$\eta_{th} = \frac{P_{th}}{A_{th}^{fuel}} < \nu$$

Eta or (thermal) fission factor: # of n emitted per thermal neutron absorbed in fuel. Typical value ~ 2.02

- Assume as fuel (F) material UO_2, with U nuclide density N^F and enrichment $\tilde{e}$

$$\eta_{th} = \frac{P_{th}}{A_{th}^{fuel}} = \frac{V^F(\nu\Sigma_{f,th}^F\phi)}{V^F(\Sigma_{a,th}^F\phi)}$$

$$\begin{cases} \Sigma_{f,th}^F = \nu\tilde{e}N^F\sigma_{f,th}^{235} \\ \Sigma_{a,th}^F = \tilde{e}N^F\sigma_{a,th}^{235} + (1-\tilde{e})N^F\sigma_{a,th}^{238} + N^O\sigma_{a,th}^O \\ = \tilde{e}N^F\sigma_{a,th}^{235} + (1-\tilde{e})N^F\sigma_{a,th}^{238} + 2N^F\sigma_{a,th}^O \end{cases}$$

$$\eta_{th} = \frac{\nu\Sigma_{f,th}^{Fuel}}{\Sigma_{a,th}^{fuel}} = \frac{\nu\tilde{e}\sigma_{f,th}^{235}}{\tilde{e}\sigma_{a,th}^{235} + (1-\tilde{e})\sigma_{a,th}^{238} + 2\sigma_{a,th}^O}$$

Depends only on fuel properties!

4 factor formula – Fuel Utilization Factor

$$f = \frac{A_{th}^{fuel}}{A_{th}} = \frac{A_{th}^{fuel}}{A_{th}^{fuel} + A_{th}^{nonFuel}}$$

(Thermal) Fuel utilization factor: fraction of thermal neutrons asorbed *in fuel.* Typical value ~ .87

- Assume mixture of fuel (F) material UO_2 and moderator (M)
- Homogenous assumption i.e. they "see" the same flux

$$f = \frac{A_{th}^{fuel}}{A_{th}} = \frac{V^F \Sigma_{a,th}^F \phi}{V^F \Sigma_{a,th}^F \phi + V^M \Sigma_{a,th}^M \phi}$$

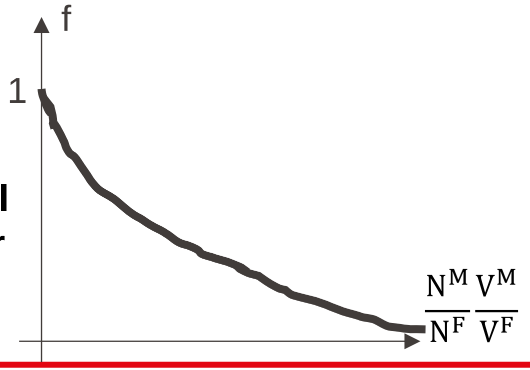
$$\Sigma_{a,th}^F = \tilde{e} N^F \sigma_{a,th}^{235} + (1 - \tilde{e}) N^F \sigma_{a,th}^{238} + 2 N^F \sigma_{a,th}^{O} = N^F \overline{\sigma_{a,th}^F}$$

(where $\overline{\sigma_{a,th}^F}$ is an effective average micro-XS for fuel)

$$\Sigma_{a,th}^{F+M} = N^F \overline{\sigma_{a,th}^F} + N^M \sigma_{a,th}^M$$

$$f = \frac{V^F \Sigma_{a,th}^F \phi}{V^F \Sigma_{a,th}^F \phi + V^M \Sigma_{a,th}^M \phi} = \frac{1}{1 + \frac{\sigma_{a,th}^M}{\sigma_{a,th}^F} \cdot \frac{N^M}{N^F} \cdot \frac{V^M}{V^F}}$$

Depends on both material properties and moderator to fuel ratio $\frac{N^M}{N^F} \frac{V^M}{V^F}$**!**



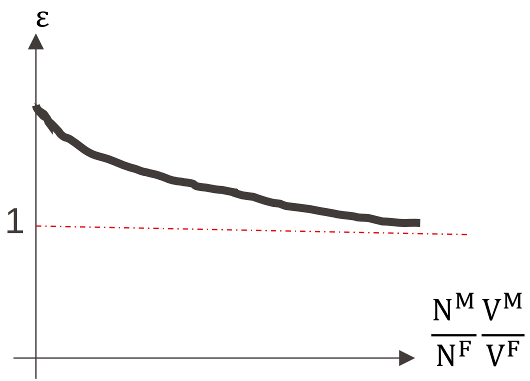
4 factor formula – Fast Fission Factor

$$\varepsilon = \frac{P}{P_{th}} = \frac{P_{th} + P_{fast}}{P_{th}} = 1 + \frac{P_{fast}}{P_{th}}$$

→ **Fast fission factor:** bonus from fast fissions in non-fissile material. Typical value ~ 1.04

$$\varepsilon = 1 + \frac{P_{fast}}{P_{th}} = 1 + \frac{\nu_{fast}^{235}\tilde{e}N^F\sigma_{f,fast}^{235} + \nu_{fast}^{238}(1-\tilde{e})N^F\sigma_{f,fast}^{238}\phi_{fast}}{\nu_{th}^{235}\tilde{e}N^F\sigma_{f,th}^{235}\phi_{th}}$$

The ratio on the right depends significantly on the moderator to fuel ratio $\frac{N^M}{N^F}\frac{V^M}{V^F}$ between 0.02 and 0.3!

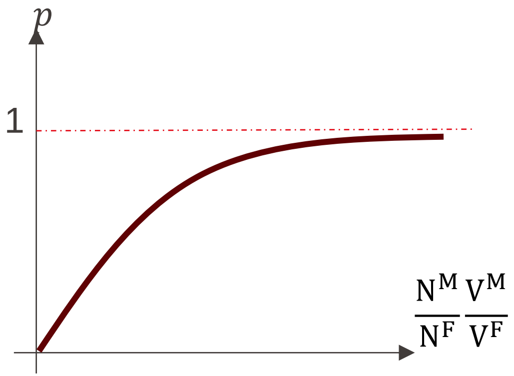


Infinite Thermal Reactor – 4 factor formula

$$p = \frac{A_{th}}{A} = \frac{A - A_{reson}}{A} = 1 - \frac{A_{reson}}{A}$$

→ **Resonance escape probability** = fraction of n absorbed i*n thermal region.* Typical value ~ 0.8

$$p \approx \exp\left(-\frac{V^F N^F (1-\tilde{e})}{V^M \xi^M \Sigma_s^M} I\right) = \exp\left(-\frac{V^F N^F (1-\tilde{e})}{N^M V^M \xi^M \sigma_s^M} I\right)$$

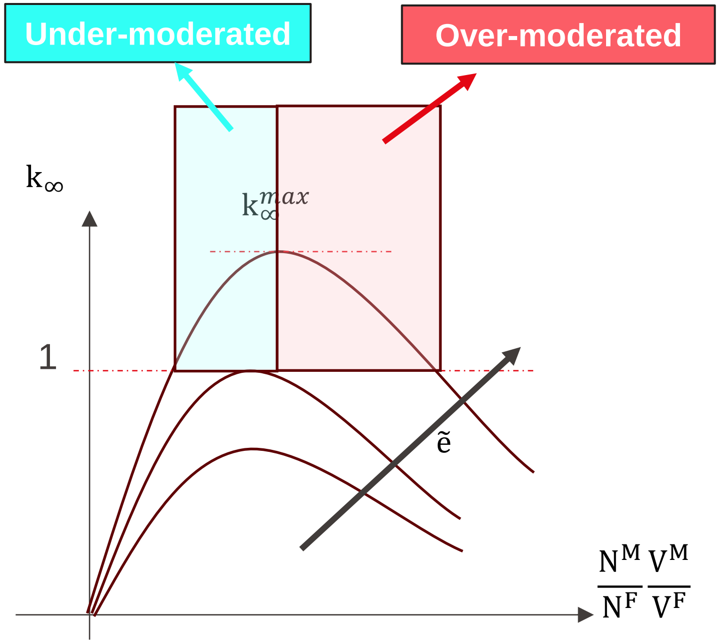


- I is the resonance integral; it describes the effectiveness of the moderator in slowing down the neutrons past the resonance region.
- Σ_s^M is the scattering cross section of the moderator.
- $N^F(1-\tilde{e})$ is the number density of the fertile material (U238).
- ξ^M is the average logarithmic energy decrease per collision (see later slides)
- p is also a function of fuel to moderator ratio → tends toward 1 as moderator ratio increases.

Infinite Thermal Reactor – 4 factor formula

- In our simplified model of a thermal reactor, we found that k_∞ is a function of material properties (XS, densities, enrichment) and of moderator to fuel ratio!

$$k_\infty = \varepsilon\left(\frac{N^M}{N^F}\frac{V^M}{V^F}\right) \times \eta_{th} \times f\left(\frac{N^M}{N^F}\frac{V^M}{V^F}\right) \times p\left(\frac{N^M}{N^F}\frac{V^M}{V^F}\right)$$



- As a function of moderator to fuel ratio, k_∞ increases, reaches a maximum and then decreases (different effects at play).
- To have a critical reactor $k_\infty > 1$, so there is a lower limit enrichment we can allow in a given reactor (that for which max k_∞ =1).
- LWR are operated typically in **under-moderated regions**: when moderator density decreases, the multiplication factor decreases.

Finite Thermal Reactor – 6 factor formula

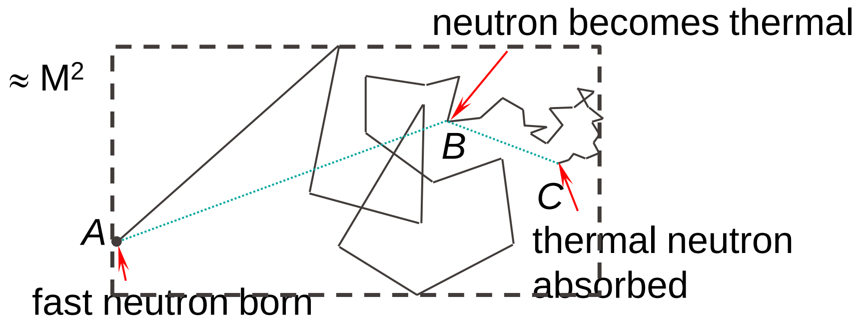
- For a reactor of finite size, one needs to consider the fraction of neutrons that leaks out:
  - Non-leakage probabilities: $\mathrm{P_f}$ (for fast neutrons), $\mathrm{P_{th}}$ (for thermal)

- Thus, the effective multiplication factor is: $\mathrm{k_{eff} = k_\infty P_f P_{th} = \varepsilon \eta_{th} f\, p P_f P_{th}}$

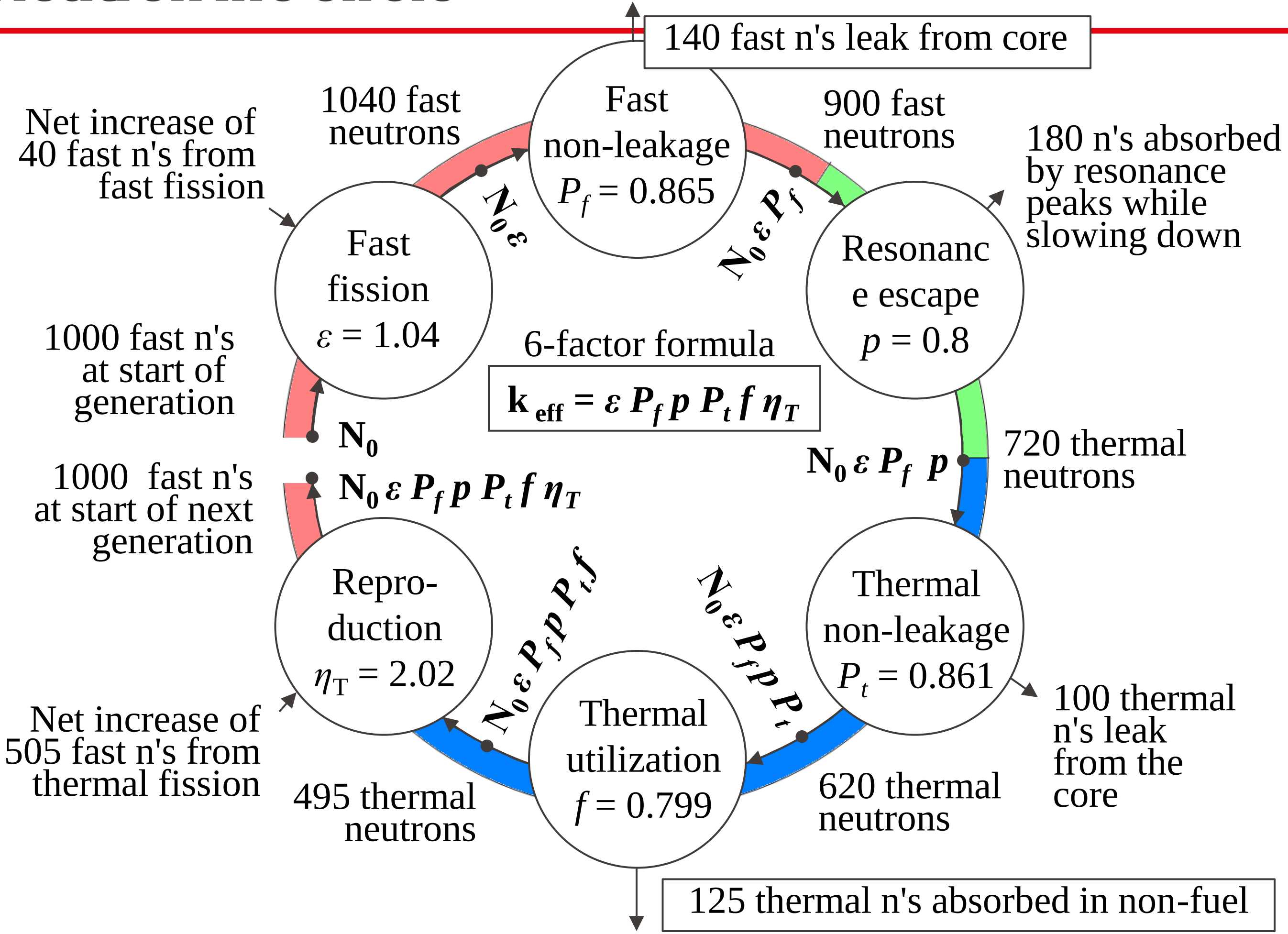
- With respect to one-group neutron balance one finds

$$P_{NF} = \mathrm{P_f P_{th}} \approx \frac{1}{1 + M^2 B^2}$$

$$k_{\mathrm{eff}} \approx \frac{k_\infty}{1 + M^2 B^2}$$



Neutron life circle



Slowing Down

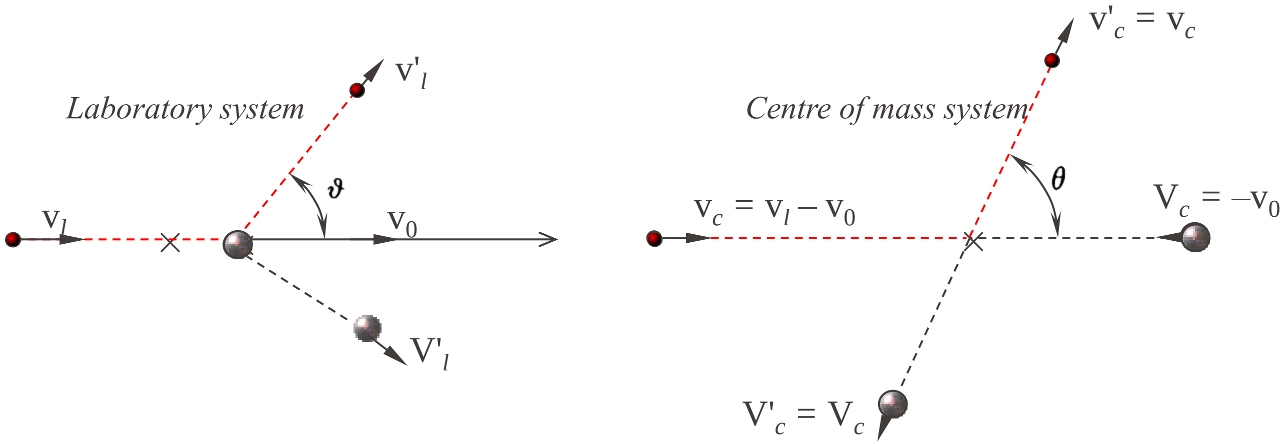
- A thermal reactor has neutrons between ~ 2 MeV and ~ 0.01 eV
  - One needs to study how neutrons change E on average across this range.
  - Slowing down process determines the “thermal -neutron source”

→ *One needs to determine the neutron energy spectrum (flux as a function of space and energy) for evaluating different reaction rates.*

→ ***In LWRs, one needs to choose a material to slow down the neutron efficiently, i.e. with as little absorption as possible.***

Study of an Elastic Collision

- Most important slowing-down mechanism: elastic scattering by moderator nuclei
  - Inelastic scattering also plays a role, but only for fast neutrons (E ≥ 1 MeV)



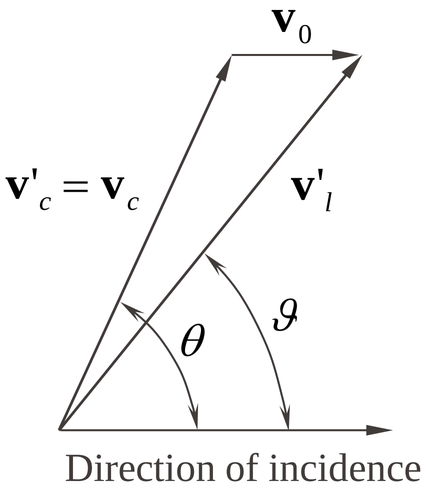
- Typically, one considers the c-mass system where the velocity before and after collision remain the same, only direction changes (derivation not discussed).

Study of an Elastic Collision - Fractional Energy Loss

Using the c-mass system one obtains (skipping derivation) the fractional energy loss as:

$$\frac{\Delta E_l}{E_l} = 1 - \frac{E'_l}{E_l} = 1 - \frac{(v'_l)^2}{(v_l)^2} = 1 - \frac{A^2 + 2A\cos\theta + 1}{(A+1)^2} = (1-\alpha)\frac{1-\cos\theta}{2}$$

where the new paramter $\alpha = \left(\frac{A-1}{A+1}\right)^2$ depends only on material mass number.



NOTE that:

- When $\theta = 0$, there is the minimum energy loss i.e. $\frac{\Delta E_l}{E_l} = 0$ or $E'_l = E_l$
- For $\theta = \pi$, there is the maximum energy loss i.e. $\frac{\Delta E_l}{E_l} = (1-\alpha)$ or $E'_l = \alpha E_l$

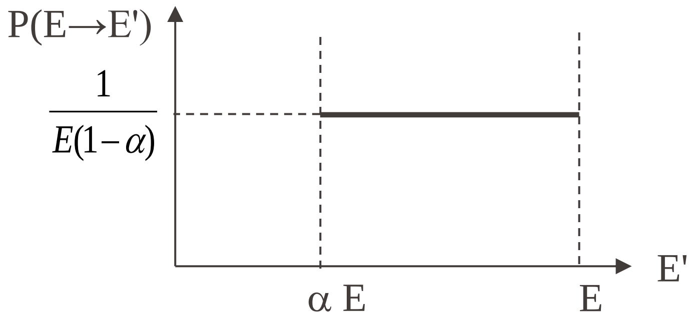
Study of an Elastic Collision - Fractional Energy Loss

- Probability distribution function P(E→E'):

  P(E→E')dE' is the probability that a neutron with the laboratory energy E will have after collision an energy between E' and E' + dE'

- In case of isotropic scattering in the center-of-mass system:

$$P(E \rightarrow E') = \begin{cases} \dfrac{1}{E(1-\alpha)} & \alpha E < E' < E \\ 0 & 0 < E' < \alpha E \end{cases}$$



- For hydrogen: $P(E \rightarrow E') = \dfrac{1}{E}$

Study of an Elastic Collision - Slow down decrement

- Average energy loss depends on material (α) and energy E

$$\langle \Delta E \rangle = \int_0^\infty (E - E') \cdot P \quad (E \to E') dE' = \int_{\alpha E}^{E} dE' \frac{E - E'}{(1-\alpha)E} = \frac{1-\alpha}{2} E$$

- One finds that **the average logarithmic energy loss per collisions (slow down decrement or ξ)** depends not on energy but only on A!

$$\xi = \int_{\alpha E}^{E} \ln\left(\frac{E}{E'}\right) P(E \to E')\, dE' \quad \Rightarrow \quad \xi = \frac{1}{E(1-\alpha)} \int_{\alpha E}^{E} \ln\left(\frac{E}{E'}\right) dE'$$

$$x = E'/E \quad \Rightarrow \quad \xi = \frac{1}{1-\alpha} \int_1^{\alpha} \ln x\, dx = 1 + \frac{\alpha}{1-\alpha} \ln \alpha$$

For A > 10 →

$$\xi \approx \frac{2}{A + \frac{2}{3}}$$

Study of an Elastic Collision - Slow down decrement

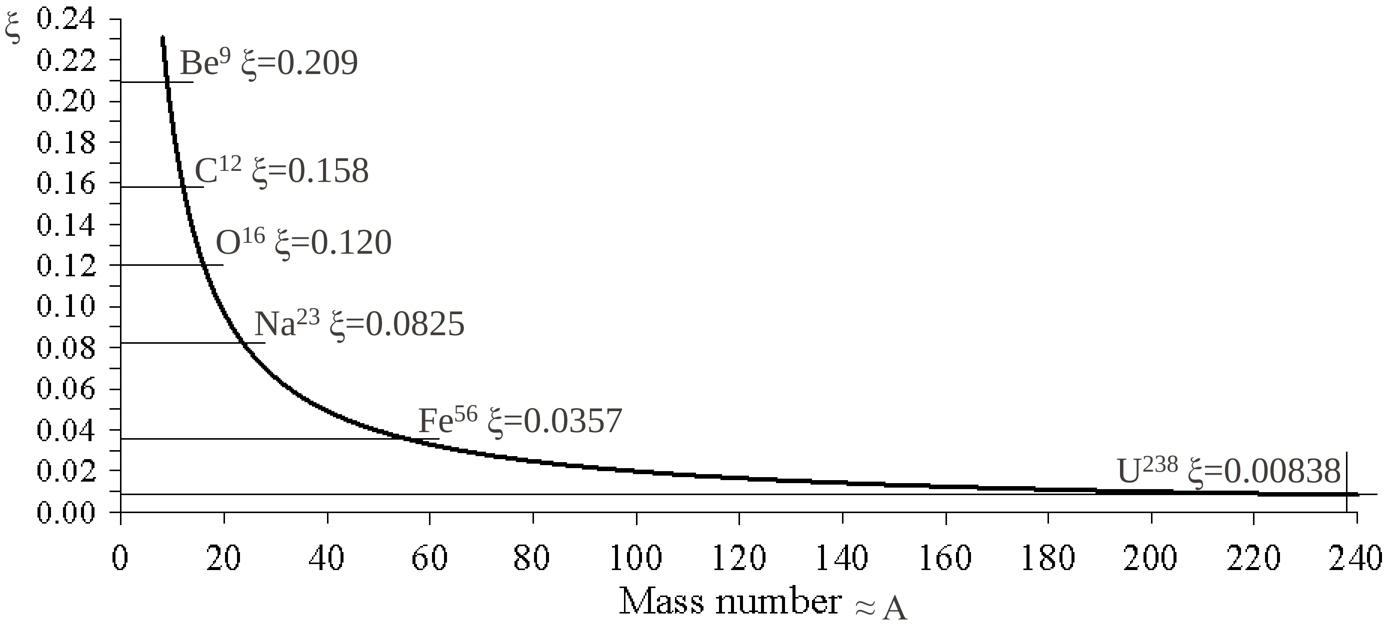
- For a mixture of nuclides:

$$\bar{\xi} = \frac{1}{\Sigma_s} \sum_i \Sigma_{si} \int_0^\infty \ln\left(\frac{E}{E'}\right) \cdot P_i\,(E \to E')dE' = \frac{1}{\Sigma_s} \sum_i \Sigma_{si} \xi_i$$

- So for instance for water:

$$\overline{\xi_{H_2O}} = \frac{\Sigma_{sH}\xi_H + \Sigma_{sO}\xi_O}{\Sigma_{sH} + \Sigma_{sO}} = \frac{2\sigma_{sH}\xi_H + \sigma_{sO}\xi_O}{2\sigma_{sH} + \sigma_{sO}}$$

Moderator characteristics



Moderator Characteristics

Not just the slow down decrement qualifies/quantifies a moderator:

- **Slow down decrement ξ** determines the minimum number of collisions to thermalize $\mathrm{N} = \ln(2\ MeV\ /\ 0.025\ eV)\ /\xi$

- A good moderator must also have Σ_s (thus its density is imporatnt). This means that the **macroscopic slowing-down power $= \xi\Sigma_s$ must be large!**

- A good moderator must also not absorb much. **Moderating Ratio $= \xi\Sigma_s/\Sigma_a$ must be large!**

TABLE 2
Moderating Properties of Materials

Material	ξ	Number of Collisions to Thermalize	Macroscopic Slowing Down Power	Moderating Ratio
H_2O	0.927	19	1.425	62
D_2O	0.510	35	0.177	4830
Helium	0.427	42	9×10^{-6}	51
Beryllium	0.207	86	0.154	126
Boron	0.171	105	0.092	0.00086
Carbon	0.158	114	0.083	216